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Orthogonal Latin hypercube designs with hidden low-dimensional projection

Tian-fang Zhang [a,b], Yue-ru Yan [a], Fasheng Sun [c,*]

[a] College of Mathematics and Statistics, Jiangxi Normal University, Nanchang 330022, China
[b] Jiangxi Provincial Center for Applied Mathematics, Nanchang 330022, China
[c] School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China

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ABSTRACT

Orthogonal Latin hypercube designs are widely used in computer experiments because of their attractive properties. In this article, we develop a new grouping method to construct such designs. Compared to the existing results, the new constructed designs can accommodate more factors with the same runsize, which means they are more cost-effective. Moreover, the resulting designs possess not only orthogonality, but also appealing space-filling properties in low dimensions, which make them very suitable for computer experiments.

1. Introduction

Computer experiments are widely used in physics and engineering for their powerful capabilities (Fang et al., 2006; Santner et al., 2018). For its one-dimensional uniformity, Latin hypercube design (McKay et al., 1979) is one of the most popular design for computer experiments. However, there may be a high degree of correlation between the factors of randomly generated Latin hypercube designs (Owen, 1994). Orthogonal Latin hypercube designs (OLHDs) (Ye, 1998) can improve the efficiency of sample selection and calculation in computer experiments. Many scientists have proposed different methods to construct orthogonal Latin hypercube designs, such as Lin et al. (2010), Steinberg and Lin (2006), Sun and Tang (2017) and so on. Among these methods, the method of rotation stands out because of its attractive properties. Steinberg and Lin (2006) gave the orthogonal Latin hypercube designs with 2^{2^u} runs by rotating 2-level orthogonal arrays. Then Pang et al. (2009) and Sun et al. (2009) extended the number of levels from 2 to p, where p is a prime power, but the run sizes must be the form of p^{2^u}. Later, Sun and Tang (2017) applied difference schemes and orthogonal arrays to fill all the run-size gaps left by the original rotation method. However, the grouping method is not flexible enough, such that the design obtained by it may not reach the maximum number of columns.

In addition to orthogonality, space-filling property is also important in computer experiments. For designs with a large number of factors of which only a few are crucial, space-filling designs with good projection properties are favorable for factor screening and model fitting. For the developments of such designs, Wang et al. (2022), Sun and Tang (2023) have done some influential work in low-dimensional projections. In order to investigate the low-dimensional stratification properties relate to other space-filling criteria, Chen and Tang (2022) studied the space-filling properties of orthogonal array-based designs in terms of a broad class of space-filling criteria. Tian and Xu (2022) proposed the space-filling criterion to classify and rank space-filling design. Tian and Xu (2024) developed a stratification pattern enumerator for evaluating stratification properties of designs.

* Corresponding author.
E-mail address: sunfs359@nenu.edu.cn (F. Sun).

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In this article, we propose an explicit grouping method to construct orthogonal Latin hypercube designs. Compared to the existing results presented by Sun and Tang (2017), our design can accommodate more factors with the same runsizes. Most importantly, the proposed designs possess not only orthogonality, but also appealing space-filling properties in low dimensions, which make them highly suitable for computer experiments.

The remainder of this paper is organized as follows. Section 2 introduces the basic notations. Section 3 presents two examples. In Section 4 we give a general approach to constructing OLHDs by rotating groups of factors in saturated orthogonal arrays and compare the results with Sun and Tang (2017). Then we study the stratification of the resulting OLHDs in Section 5. Some discussions are given in Section 6. All the proofs and some tables are given in Appendix.

2. Preliminary and notation

A Latin hypercube design is an $n\times m$ matrix which each column is a permutations of the levels taken from $\Omega(n) = \{-(n-1)/2, -(n-2)/2, \dots, (n-1)/2\}$, and it becomes an orthogonal Latin hypercube if any two columns are orthogonal. We use $OLHD(n, m)$ to denote such an orthogonal Latin hypercube with n runs and m factors.

An orthogonal array is of strength t with n runs, k factors and p levels, denoted as $OA(n, k, p, t)$, is an $n\times k$ matrix with entries from a set of p levels such that all the t-tuples occur equally often in each subarray of t columns. Especially, for $t = 2$, if $k = (n-1)/(p-1)$, then the $OA(n, k, p, 2)$ is a saturated design. The factors of a design $OA(n, k, p, t)$, also called columns, are denoted by $1, 2, \dots, k$. A very popular subclass of orthogonal arrays, referred to as fractions with a defining relation, or regular fractions, has very useful in our paper. It should be mentioned that the orthogonal arrays used in our construction are regular if not specified otherwise.

Alternatively, the saturated orthogonal array $OA(n, k, p, 2)$ with $n = p^d$ can be represented by some elements of $GF(p^d)$, for $d = 2^u$ and $u = 1, 2, \dots$. Let $GF(p) = \{0, 1, \dots, p-1\}$, where p is a prime. Consider $GF(p^d) = \{a_0 + a_1x + \cdots + a_{d-1}x^{d-1},\ a_i \in GF(p)\}$, with the primitive polynomial $f(x) = c_0 + c_1x + \cdots + c_dx^d$, where x is a primitive element and $c_i \in GF(p)$, $i = 0, 1, \dots, d$. Then the p^d elements of $GF(p^d)$ can be represented as $GF(p^d) = \{x^0, x^1, \dots, x^{p^d-2}, 0\}$ since x is a primitive element. By taking the first k elements, $x^0, x^1, \dots, x^{k-1}$, we can obtain k different formulas modulo $f(x)$. Such k elements correspond to the columns of the orthogonal array $OA(n, k, p, 2)$. For example, let $GF(3^2) = \{a_0 + a_1x, a_0, a_1 \in GF(3)\}$ with $GF(3) = \{0, 1, 2\}$ and $f(x) = x^2 + x + 2$ be a primitive polynomial. Then x^0, x^1, x^2, x^3 modulo $f(x)$, obtaining $1, x, 1+2x, 2+2x$ which correspond to the columns $1, 2, 12^2, 1^22^2$. In general, the element $a_0 + a_1x + \cdots + a_{d-1}x^{d-1}$ is associated with the generalized interaction column $1^{a_0}2^{a_1}\cdots d^{a_{d-1}}$ of all factors i for which $a_{i-1} \neq 0$.

Pang et al. (2009) obtained the following result, we show it below.

Lemma 1. *The first k nonzero elements of $GF(p^d)$ corresponding to the powers of x, i.e., $x^0, x^1, \dots, x^{k-1}$, provide an ordering of the effect columns in matrix D. The sets consisting of d consecutive columns in order are full factorial designs $D_1, \dots, D_b$ in sequence, where $b = \lfloor k/d \rfloor$.*

After converting the levels of an orthogonal array into $\{(2i - p - 1)/2,\ i = 1, 2, \dots, p\}$ for odd p, or $\{2i - p - 1,\ i = 1, 2, \dots, p\}$ for even p, such orthogonal array can be rotated into an OLHD. Therefore the rotation matrix is important for constructing OLHDs. Let us review some notations of the rotation matrix. It is well known that a $d\times d$ real matrix R is called a rotation matrix if $R^TR = cI_d$, where I_d is a $d\times d$ identity matrix and c is a constant. Now we will introduce two types of rotation matrices.

First, let

$$R_{10} = \begin{bmatrix} p & -1 \\ 1 & p \end{bmatrix},\ R_{u0} = \begin{bmatrix} p^{2^{(u-1)}}R_{(u-1)0} & -R_{(u-1)0} \\ R_{(u-1)0} & p^{2^{(u-1)}}R_{(u-1)0} \end{bmatrix},\ u \geq 2,$$

and

$$Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ Q_u = \begin{bmatrix} Q_{u-1} & 0 \\ 0 & -Q_{u-1} \end{bmatrix},\ u \geq 2.$$

Based on them, the first type of rotation matrices taken from Sun and Tang (2017), is defined as follows:

$$R_{u1} = \begin{bmatrix} pR_{u0} & -Q_u \\ Q_u & pR_{u0} \end{bmatrix},\ R_{uv} = \begin{bmatrix} pR_{u(v-1)} & -Q_{u+v-1} \\ Q_{u+v-1} & pR_{u(v-1)} \end{bmatrix},\ v \geq 2. \tag{1}$$

Next we define a new type of rotation matrices. Let $u = v = 1$ in Eq. (1), we can get R_{11},

$$R_{11} = \begin{bmatrix} p^2 & -p & -1 & 0 \\ p & p^2 & 0 & 1 \\ 1 & 0 & p^2 & -p \\ 0 & -1 & p & p^2 \end{bmatrix}.$$

Then, let

$$S_{20} = R_{11},\ \ V_u = \begin{bmatrix} (p^3)^{2^{(u-1)}} & -1 \\ 1 & (p^3)^{2^{(u-1)}} \end{bmatrix},\ u \geq 1,$$

we obtain the following matrices

$$S_{30} = V_1 \otimes S_{20} = \begin{bmatrix} p^3 S_{20} & -S_{20} \\ S_{20} & p^3 S_{20} \end{bmatrix}, \tag{2}$$

$$S_{u0} = V_{(u-2)} \otimes S_{(u-1)0} = \begin{bmatrix} (p^3)^{2^{u-3}} S_{(u-1)0} & -S_{(u-1)0} \\ S_{(u-1)0} & (p^3)^{2^{u-3}} S_{(u-1)0} \end{bmatrix}, \ u \geq 3. \tag{3}$$

According to Q_u and S_{u0}, the new matrix S_{uv} can be constructed as follows:

$$S_{u1} = \begin{bmatrix} pS_{u0} & -Q_u \\ Q_u & pS_{u0} \end{bmatrix}, \ S_{uv} = \begin{bmatrix} pS_{u(v-1)} & -Q_{u+v-1} \\ Q_{u+v-1} & pS_{u(v-1)} \end{bmatrix}, \ v \geq 2. \tag{4}$$

It is obvious that the order of S_{uv} is 2^{u+v}. And the nonzero entries in each column of S_{uv} are a signed permutation of $1, p, p^2, \ldots, p^{3\cdot 2^{u-2}+v-1}$, for $u \geq 2, v \geq 0$.

Lemma 2. *The matrix S_{uv} is a rotation matrix.*

The matrices S_{uv} $(u \geq 2, v \geq 0)$ are called the second type of rotation matrices.

3. Two examples

In order to illustrate our main ideas, we present two examples which are constructed by different grouping methods and rotation matrices. These two examples will generate an $OLHD(64, 57)$ and an $OLHD(243, 104)$, which can accommodate an additional 9 and 24 columns than those of Sun and Tang (2017) respectively.

Example 1 (*Construction of an $OLHD(64, 57)$.*)**.** Consider the following seven disjoint groups with two parts each from an $OA(64, 63, 2, 2)$.

$$\begin{aligned} D_1 &= (1, 2, 3, 123 \ \vdots \ 4, 5, 6, 456), \\ D_2 &= (12, 23, 34, 14 \ \vdots \ 45, 56, 126, 124), \\ D_3 &= (13, 24, 35, 1245 \ \vdots \ 46, 125, 236, 1345), \\ D_4 &= (1234, 2345, 3456, 1346 \ \vdots \ 12456, 1356, 146, 1236), \\ D_5 &= (15, 26, 235, 136 \ \vdots \ 234, 345, 2356, 36), \\ D_6 &= (1256, 346, 2456, 12356 \ \vdots \ 25, 12346, 145, 256), \\ D_7 &= (134, 356, 1235, 2346 \ \vdots \ 1246, 246, 23456, 123456). \end{aligned} \tag{5}$$

Each group satisfies the conditions: (i) every part with four columns is an orthogonal array of strength 3; (ii) if we delete one column in each part, the remaining columns of each group can form a full factorial design. Take D_1 as an example, both the first part 1,2,3,123 and the second part $(4, 5, 6, 456)$ is an $OA(64, 4, 2, 3)$. If we delete 123 in the first part and 456 in the second part, the remaining columns 1,2,3,4,5,6 form a 2^6 full factorial design. Actually one can delete any column in the first part and any column in the second part, such that the remaining six columns form a 2^6 full factorial design. After centralizing $D_1, D_2, \ldots, D_7$, we can get $D_1^*, D_2^*, \ldots, D_7^*$. Then, rotating them by the following rotation matrix

$$S = \begin{pmatrix} 32 & -16 & -8 & 0 & -4 & 2 & 1 & 0 \\ 16 & 32 & 0 & 8 & -2 & -4 & 0 & -1 \\ 8 & 0 & 32 & -16 & -1 & 0 & -4 & 2 \\ 0 & -8 & 16 & 32 & 0 & 1 & -2 & -4 \\ 4 & -2 & -1 & 0 & 32 & -16 & -8 & 0 \\ 2 & 4 & 0 & 1 & 16 & 32 & 0 & 8 \\ 1 & 0 & 4 & -2 & 8 & 0 & 32 & -16 \\ 0 & -1 & 2 & 4 & 0 & -8 & 16 & 32 \end{pmatrix},$$

we can obtain an $OLHD(64, 56)$, denoted by $(D_1^*S \ \vdots \ D_2^*S \ \vdots \ \cdots \ \vdots \ D_7^*S)$. Note that after grouping, there are seven columns left in $OA(64, 63, 2, 2)$, i.e., 245, 135, 12345, 13456, 1456, 156,16. The first six columns are orthogonal to all other columns and form a full factorial design which can create one more orthogonal column of 2^6-level. So an $OLHD(64, 57)$ can be constructed. The number of columns is 9 more than that of Sun and Tang (2017). The resulting $OLHD(64, 57)$ is displayed in Table 3 in Appendix.

Example 2 (*Construction of An $OLHD(243, 104)$.*)**.** Consider a saturated orthogonal array $OA(81, 40, 3, 2)$. Divide such array into 10 groups with four independent columns each by Lemma 1. The groups are shown in the following table.

Group	Column			
D_1	1	2	3	4
D_2	14^2	1^224	12^234^2	$1^223^24^2$
D_3	1^22^23	2^23^24	13^24	124
D_4	1234^2	1^2234^2	$1^22^234^2$	$1^22^23^24^2$
D_5	$1^22^23^2$	$2^23^24^2$	1^23^2	2^24^2
D_6	1^23^24	12^24	123^24^2	1^223
D_7	2^234	13^2	24^2	1^234
D_8	12^2	23^2	34^2	1^24^2
D_9	1^22^24	$12^23^24^2$	1^223^2	2^234^2
D_{10}	$1^23^24^2$	1^22^2	2^23^2	3^24^2

Then add the fifth independent column into the above table, we have

5	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	D_{10}
5^0	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	D_{10}
5^1	$5D_1$	$5D_2$	$5D_3$	$5D_4$	$5D_5$	$5D_6$	$5D_7$	$5D_8$	$5D_9$	$5D_{10}$
5^2	5^2D_1	5^2D_2	5^2D_3	5^2D_4	5^2D_5	5^2D_6	5^2D_7	5^2D_8	5^2D_9	5^2D_{10}

Any five columns, obtained by taking four columns from 5^aD_i and one column from 5^bD_i with $a \neq b$ $(a, b = 0, 1, 2)$, form a 3^5 full factorial design. The ten groups are shown as follows.

Group	Group
$D_{1,1} = (5D_1 \ \vdots\ 5^2D_1)$	$D_{6,1} = (D_6 \ \vdots\ 5D_6)$
$D_{2,1} = (5D_2 \ \vdots\ 5^2D_2)$	$D_{7,1} = (D_7 \ \vdots\ 5^2D_7)$
$D_{3,1} = (D_3 \ \vdots\ 5^2D_3)$	$D_{8,1} = (D_8 \ \vdots\ 5D_8)$
$D_{4,1} = (D_4 \ \vdots\ 5D_4)$	$D_{9,1} = (5D_9 \ \vdots\ 5^2D_9)$
$D_{5,1} = (D_5 \ \vdots\ 5D_5)$	$D_{10,1} = (D_{10} \ \vdots\ 5^2D_{10})$

After dividing, there are the column 5 and groups $D_1, D_2, 5D_3, 5^2D_4, 5^2D_5, 5^2D_6, 5D_7$, 5^2D_8, D_9, 5^2D_{10} left. Among the remaining, three new groups which satisfy the above property can be formed. For convenience, we mark them as

$$D_{11,1} = (1, 2, 3, 4 \ \vdots\ 5, 1^22^23^24^25^2, 1^2234^25^2, 1^22^25^2),$$
$$D_{11,2} = (14^2, 1^224, 12^234^2, 1^223^24^2 \ \vdots\ 1^22^234^25^2, 13^25, 1^23^25^2, 13^245),$$
$$D_{11,3} = (1^22^24, 12^23^24^2, 1^223^2, 2^234^2 \ \vdots\ 1^23^24^25^2, 2^23^25^2, 2^2345, 24^25).$$

As a result, we can get 13 groups in total. Centralizing them, $D^*_{1,1}, \ldots, D^*_{10,1}, D^*_{11,1}, \ldots, D^*_{11,3}$ are obtained. Then, each of them is rotated by

$$R = \begin{pmatrix} 81 & -27 & -9 & 3 & -1 & 0 & 0 & 0 \\ 27 & 81 & -3 & -9 & 0 & 1 & 0 & 0 \\ 9 & -3 & 81 & -27 & 0 & 0 & 1 & 0 \\ 3 & 9 & 27 & 81 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 81 & -27 & -9 & 3 \\ 0 & -1 & 0 & 0 & 27 & 81 & -3 & -9 \\ 0 & 0 & -1 & 0 & 9 & -3 & 81 & -27 \\ 0 & 0 & 0 & 1 & 3 & 9 & 27 & 81 \end{pmatrix}.$$

An $OLHD(243, 104)$, $D^*R = (D^*_{1,1}R \ \vdots\ \cdots \ \vdots\ D^*_{10,1}R \ \vdots\ D^*_{11,1}R \ \vdots\ \cdots \ \vdots\ D^*_{11,3}R)$, is obtained here. The number of columns is 27 more than that of Sun and Tang (2017).

4. General results

4.1. Grouping method

Let $D = (D_1, D_2, \ldots, D_b)$, where all the D_is are disjoint subarrays of an saturated orthogonal array $OA(p^d, k, p, 2)$. To construct OLHDs by the first or second type of the rotation matrices R_{uv} in (1) or S_{uv} in (4), we should construct an orthogonal array whose columns are grouped into subarrays with 2^{u+v} columns. From Lemma 1, the grouping subarray that uses R_{uv} must satisfy the following condition.

Condition 1. *Each D_i must be a p^d full factorial design, for $d = 2^u, u \geq 2$.*

To construct OLHDs by the second type of the rotation matrices S_{uv} in (4), the subarray grouping must satisfy the following condition.

Condition 2. *Let*

$$D_i = (B_{i,1}, B_{i,2}, \dots, B_{i,2^{u-2}}), \quad 1 \le b \le \lfloor k/d \rfloor, \tag{6}$$

then each $B_{i,j}$ $(j = 1, \dots, 2^{u-2})$ *with four columns must be an orthogonal array of strength 3 and if one column in each* $B_{i,j}$ *is deleted, the remaining columns of* D_i *must form a* p^d *full factorial design, for* $d = 3 \cdot 2^u, u \ge 2$.

$D_1, D_2, \dots, D_b$ satisfying Condition 1 or Condition 2 can be seen as the original groups to construct OLHDs using R_{uv} or S_{uv} separately.

Now we consider the groups of $OA(p^{d+1}, (p^{d+1}-1)/(p-1), p, 2)$. Let $D_1, D_2, \dots, D_b$ be the groups of $OA(p^d, (p^d-1)/(p-1), p, 2)$ by Lemma 1. By adding a new dependent column $d+1$ into each D_i, we have

$d+1$	D_1	D_2	$\cdots$	D_b
$(d+1)^0$	$(d+1)^0 D_1$	$(d+1)^0 D_2$	$\cdots$	$(d+1)^0 D_b$
$(d+1)^1$	$(d+1)^1 D_1$	$(d+1)^1 D_2$	$\cdots$	$(d+1)^1 D_b$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(d+1)^{p-1}$	$(d+1)^{p-1} D_1$	$(d+1)^{p-1} D_2$	$\cdots$	$(d+1)^{p-1} D_b$

Based on the property of D_i, we have the following result.

Proposition 1. *Consider a new group* $((d+1)^{a_t} D_i \ \vdots \ (d+1)^{a_k} D_i)$, $a_t, a_k = 0, 1, \dots, p-1$ *and* $a_t \ne a_k$. *Suppose* α *is any column of* $(d+1)^{a_k} D_i$. *We have*

(i) if D_i *satisfies* Condition 1, *then* $((d+1)^{a_t} D_i \ \vdots \ \alpha)$ *is a* p^{d+1} *full factorial design.*
(ii) if D_i *satisfies* Condition 2, *then* α *and the remaining columns of* $(d+1)^{a_t} D_i$ *after deleting one column in each* $(d+1)^{a_t} B_{ij}$, *for* $j = 1, 2, \dots, 2^{u-2}$, *form a* p^{d+1} *full factorial design.*

The proof is postponed to Appendix..

Now the groups of $OA(p^{d+1}, (p^{d+1}-1)/(p-1), p, 2)$ can be obtained:

(i) when p is even, for each i, group D_i can generate $p/2$ groups of $OA(p^{d+1}, (p^{d+1}-1)/(p-1), p, 2)$

$$\begin{aligned} D_{i,1} &= ((d+1)^{a_0} D_i \ \vdots \ (d+1)^{a_1} D_i), \\ &\cdots \\ D_{i,p/2} &= ((d+1)^{a_{p-2}} D_i \ \vdots \ (d+1)^{a_{p-1}} D_i), \end{aligned} \tag{7}$$

where $a_0, \dots, a_{p-1}$ are different integers of $\{0, 1, \dots, p-1\}$. Since $i = 1, 2, \dots, b$, there are $s_1 = pb/2$ groups in total.

(ii) when p is odd, for each D_i, in addition to the above $\lfloor p/2 \rfloor$ groups in (7), there is always remaining one group for each i. So there are column $d+1$ and b groups left. Find some proper columns from all the remaining columns to form new groups (suppose there are t_1 groups) satisfying Proposition 1, denoted such groups as

$$D_{b+1,1}, D_{b+1,2}, \dots, D_{b+1,t_1}.$$

Then we can get $s_1 = b\lfloor p/2 \rfloor + t_1$ groups in all, i.e.,

$$D_{1,1}, D_{1,2}, \dots, D_{1,\lfloor p/2 \rfloor}, \dots, D_{b,1}, \dots, D_{b,\lfloor p/2 \rfloor}, D_{b+1,1}, D_{b+1,2}, \dots, D_{b+1,t_1}. \tag{8}$$

Repeat this procedure, the groups of $OA(p^{d+v}, (p^{d+v}-1)/(p-1), p, 2)$ can be obtained.

4.2. Construct OLHDs

In this section, we propose an algorithm to construct OLHDs.

Algorithm

step 1. Find b groups of $OA(p^d, (p^d-1)/(p-1), p, 2)$ satisfying Condition 1 by Lemma 1 or Condition 2 by searching method, i.e., $D_1, D_2, \dots, D_b$.

step 2. Add a new dependent column $d+1$ into $D_1, D_2, \dots, D_b$, some groups of $OA(p^{d+1}, (p^{d+1}-1)/(p-1), p, 2)$ are obtained, i.e., $D_{i,1}, \dots, D_{i,\lfloor p/2 \rfloor}$. Find additional groups satisfying Condition 1 or Condition 2, obtain $D_{b+1,1}, D_{b+1,2}, \dots, D_{b+1,t_1}$. Then we can get $s_1 = b\lfloor p/2 \rfloor + t_1$ groups in total.

step 3. Repeat Step 2 v times. Groups $D_{i,i_1 \cdots i_v}$ satisfying Proposition 1 can be obtained.

(i) For even p, $i = 1, 2, \dots, b$, $i_j = 1, 2, \dots, p/2$, $j = 1, 2, \dots, v$. There are $s_v = s_{v-1} p/2$ groups.

(ii) For odd p, $i = 1, 2, \dots, b$, $i_j = 1, 2, \dots, \lfloor p/2 \rfloor$, $j = 1, 2, \dots, v$; when $i = b+1$, then $i_j = 1, 2, \dots, t_j$. There are $s_v = s_{v-1}\lfloor p/2 \rfloor + t_v$ groups. Denote such s_v groups as

$$D = (D^{(1)}, \dots, D^{(s_v)}). \tag{9}$$

Table 1
Some constructed OLHDs and comparisons.

p	d	n	m	Rotation Matrix	Source	π
2	5	32	24 (24)	R_{21}	Theorem 1	69.56%
2	6	64	57 (48)	S_{30}	Example 1	94.55%
2	7	128	113 (96)	S_{31}	Theorem 2	93.69%
3	3	27	12 (12)	R_{11}	Theorem 1	72.73%
3	5	243	104 (80)	R_{21}	Example 2	97.09%
3	6	729	208 (160)	R_{22}	Theorem 1	96.62%
3	7	2187	416 (320)	R_{23}	Theorem 1	96.39%

step 4. Centralize each $D^{(i)}$ and get $D^{(i)*}$. Then the resulting design

$$Z^* = (Z_1^*, \dots, Z_{s_v}^*) \tag{10}$$

can be obtained by rotating each $D^{(i)*}$, where the matrix $Z_i^* = D^{(i)*}P$ and P is defined in (1) or (4).

In step 1, we can get the groups satisfying Condition 1 according to Lemma 1. Unfortunately, so far no general method has been found to construct groupings satisfying Condition 2.

Using Lemma 1 and the rotation matrices R_{uv}, we have the following theorem.

Theorem 1. *Let $D = (D_1, D_2, \dots, D_b)$ be an orthogonal array with p^d $(d = 2^u, u \geq 1)$ runs such that each D_i is a p^d full factorial design and $b = \lfloor k/d \rfloor$, then an $OLHD(p^{d+v}, m)$ can be constructed, where (i) $m = b2^u p^v$ for even p, (ii) $m \geq b\lfloor p/2 \rfloor^v 2^{u+v}$ for odd p.*

The proof is shown in Appendix.

Remark 1. For even p, the result of Theorem 1 can be found in Sun and Tang (2017). However, if we can find any other new group in the remaining columns for odd p, our result will be better than that of Sun and Tang (2017), such as Example 2.

By the rotation matrices S_{uv} and Proposition 1, we have the following result.

Theorem 2. *Let $D = (D_1, D_2, \dots, D_b)$ be an orthogonal array with $p^d (d = 3 \cdot 2^{u-2}, u \geq 2)$ runs and each D_i satisfy Condition 2, then an $OLHD(p^{3\cdot 2^{u-2}+v}, b\lfloor p/2 \rfloor^v 2^{u+v})$ can be constructed, where $b \leq \lfloor (p^d - 1)/((p-1)2^u) \rfloor$.*

The proof is similar to Theorem 1, we omit it here.

Example 1 is the case for $p = 2$, $u = 3$ in Theorem 2. In order to make the construction method more clear, we give another example.

Example 3. Following the grouping (5) in Example 1, we add a new independent column 7 in it. By Step 2 of the above algorithm, the new table is obtained.

7	D_1	D_2	D_3	D_4	D_5	D_6	D_7
7^0	D_1	D_2	D_3	D_4	D_5	D_6	D_6
7^1	$7D_1$	$7D_2$	$7D_3$	$7D_4$	$7D_5$	$7D_6$	$7D_7$

Then we have seven new groups, i.e., $D_{1,1} = (D_1 \ \vdots \ 7D_1), D_{2,1} = (D_2 \ \vdots \ 7D_2), \dots, D_{7,1} = (D_7 \ \vdots \ 7D_7)$. Centralizing them, we get $D_{1,1}^*, D_{2,1}^*, \dots, D_{7,1}^*$. For $i = 1, 2, \dots, 7$, rotate each $D_{i,1}^*$ by the rotation matrix S_{31}. An $OLHD(128, 112)$ can be obtained by $(D_{1,1}{}^* S_{31} \ \vdots \ D_{2,1}{}^* S_{31} \ \vdots \ \cdots \ \vdots \ D_{7,1}{}^* S_{31})$. After grouping, there are 15 columns left. The seven columns $245, 135, 12345, 13456, 1456, 156, 7$ can create one more orthogonal column of 2^7-level. So an $OLHD(128, 113)$ can be constructed, which is 17 columns more than the design by Sun and Tang (2017).

4.3. Comparison

We compare some of the results with that of Sun and Tang (2017) in Table 1. The column m is gotten by our method. Results in bracket are from Sun and Tang (2017). The values of π from Wang et al. (2021) can characterize the space-filling properties, where $\pi = (\kappa - 1)\mu/(\kappa\mu - 1)$, κ is the number of groups and μ is the number of columns in each group. When the values of π are close to 1, it means that nearly all column pairs achieve stratifications on $p \times p^2$ grids.

5. Stratification

From Table 1, most of the values of π are very close to 1, which means that nearly all column pairs achieve stratifications on $p \times p^2$ grids. In this section, we will study the general space-filling properties of the constructed $OLHD(p^{d+v}, m)$ in Section 4. For an array with n runs and m columns, we say it achieve a stratification on a $p_1 \times p_2 \times \cdots \times p_r$ grid in some r $(r \geq 2)$ dimensions if the corresponding r columns of it can be collapsed into an $OA(n, r, p_1 \times p_2 \times \cdots \times p_r, r)$.

Based on the form of the rotation matrix R_{uv} (or S_{uv}), we know any column of the constructed $OLHD(p^{d+v}, m)$ is the signed permutation of $1, p, p^2, \ldots, p^{d+v-1}$. Therefore, we can assume any column x in $OLHD(p^{d+v}, m)$ has the following form:

$$x = p^{d+v-1}e^{(1)} \pm p^{d+v-2}e^{(2)} \pm \cdots \pm pe^{(d+v-1)} \pm e^{(d+v)} \tag{11}$$

where $e^{(1)}, e^{(2)}, \ldots, e^{(d+v-1)}$ are the columns of D in (9). Now we consider the mapping

$$f^{(i)}(z) = \lfloor \frac{z + (p^{d+v} - 1)/2}{p^{d+v-i}} \rfloor - (p-1)/2, \quad for\ z \in \Omega(p^{d+v}), \tag{12}$$

which collapses the p^{d+v} levels in $\Omega(p^{d+v})$ into p^i levels. For example, when $p = 3$, $d = 4$, $v = 1$ and $i = 1$, the 243 levels are collapsed into 3 levels by the mapping $f^{(1)}$ as follows:

$$\begin{array}{ccccccc} -121, & -120, & \cdots & -42, & -41 & \rightarrow & -1 \\ -40, & -39, & \cdots & 39, & 40 & \rightarrow & 0 \\ 41, & 42, & \cdots & 120, & 121 & \rightarrow & 1 \end{array}$$

Consider two columns d_1 and d_2 of Z^*. By (11) they can be written in the following form

$$\begin{aligned} d_1 &= p^{d+v-1}e_1^{(1)} \pm p^{d+v-2}e_1^{(2)} \pm \cdots \pm pe_1^{(d+v-1)} \pm e_1^{(d+v)} \\ d_2 &= p^{d+v-1}e_2^{(1)} \pm p^{d+v-2}e_2^{(2)} \pm \cdots \pm pe_2^{(d+v-1)} \pm e_2^{(d+v)}. \end{aligned} \tag{13}$$

Now we collapse the p^{d+v} levels of d_1 and d_2 in $\Omega(p^{d+v})$ into p^i and p^j levels by the mapping (11) respectively. We have

$$\begin{aligned} f^{(i)}(d_1) &= p^{i-1}e_1^{(1)} \pm p^{i-2}e_1^{(2)} \pm \cdots \pm e_1^{(i)}, \\ f^{(j)}(d_2) &= p^{j-1}e_2^{(1)} \pm p^{j-2}e_2^{(2)} \pm \cdots \pm e_2^{(j)}. \end{aligned} \tag{14}$$

If $(f^{(i)}(d_1), f^{(j)}(d_2))$ is an $OA(p^{d+v}, 2, p^i \times p^j, 2)$, then (d_1, d_2) can achieve a $p^i \times p^j$ grid. Such stratification can be achieved if $(e_1^{(1)}, \ldots, e_1^{(i)}, e_2^{(1)}, \ldots, e_2^{(j)})$ forms an orthogonal array of strength $i + j$. This holds just for the property of the independent columns.

By the above discussion, we have the following lemma.

Lemma 3. *If $(e_1^{(1)}, \ldots, e_1^{(i)}, e_2^{(1)}, \ldots, e_2^{(j)})$ is an $OA(p^{d+v}, i+j, p, i+j)$, then (d_1, d_2) in (13) can achieve stratification on a $p^i \times p^j$ grid.*

Furthermore, consider three columns d_1, d_2 and d_3 of Z^*, where d_1 and d_2 have the form of (13), and d_3 has the following form

$$d_3 = p^{d+v-1}e_3^{(1)} \pm p^{d+v-2}e_3^{(2)} \pm \cdots \pm pe_3^{(d+v-1)} \pm e_3^{(d+v)}. \tag{15}$$

We collapse the p^{d+v} levels of d_3 in $\Omega(p^{d+v})$ into p^k levels by the mapping (12)

$$f^{(k)}(d_3) = p^{k-1}e_3^{(1)} \pm p^{k-2}e_3^{(2)} \pm \cdots \pm e_3^{(k)}. \tag{16}$$

Similar to Lemma 3, we have the following result.

Lemma 4. *If $(e_1^{(1)}, \ldots, e_1^{(i)}, e_2^{(1)}, \ldots, e_2^{(j)}, e_3^{(1)}, \ldots, e_3^{(k)})$ is an $OA(p^{d+v}, i+j+k, p, i+j+k)$, then (d_1, d_2, d_3) can achieve stratification on a $p^i \times p^j \times p^k$ grid.*

Based on the above discussion, we give the theorem about the stratification of the resulting $OLHD(p^{d+v}, m)$.

Theorem 3. *The design Z^* in (10) obtained by R_{uv} or S_{uv} has the following properties:*

(i) any two distinct columns achieve a stratification on a $p \times p$ grid;
(ii) any three distinct columns from the same group Z_i^* achieve a stratification on a $p \times p \times p$ grid;
(iii) $2^{u+v}(2^{u+v-1} - 1)$ column pairs from the same group Z_i^* achieve a stratification on a $p \times p^2$ grid or a $p^2 \times p$ grid.
(iv) $2^{u+v}(2^{u+v-1} - 1)$ column pairs by R_{uv} from the same group Z_i^* achieve a stratification on a $p^2 \times p^2$ grid.

R_{uv} here is as in Sun and Tang (2017), so the space-filling properties of the OLHDs gotten by R_{uv} are identical to those of the OLHDs constructed by Sun and Tang (2017). Since S_{uv} is the new type rotation matrix, we show some additional space-filling properties of the OLHDs constructed by S_{uv} as follows.

Theorem 4. *The design Z^* in (10) obtained by S_{uv} has $2^{u+v+1}(2^{u+v-2} - 1)$ column pairs achieving a stratification on a $p^2 \times p^2$ or a $p^3 \times p^3$ grid in the same group.*

Table 2
The stratifications of the first 8 columns of the $OLHD(64, 57)$.

Stratification	Column pairs
2×4 or 4×2	(1,3),(1,4),(1,5),(1,6),(1,7),(1,8),(2,3),(2,4),(2,5),(2,6),(2,7),(2,8),(3,5),(3,6),(3,7),(3,8),(4,5),(4,6),(4,7),(4,8),(5,7),(5,8),(6,7),(6,8)
4×4 or 8×8	(1,5),(1,6),(1,7),(1,8),(2,5),(2,6),(2,7),(2,8),(3,5),(3,6),(3,7),(3,8),(4,5),(4,6),(4,7),(4,8).

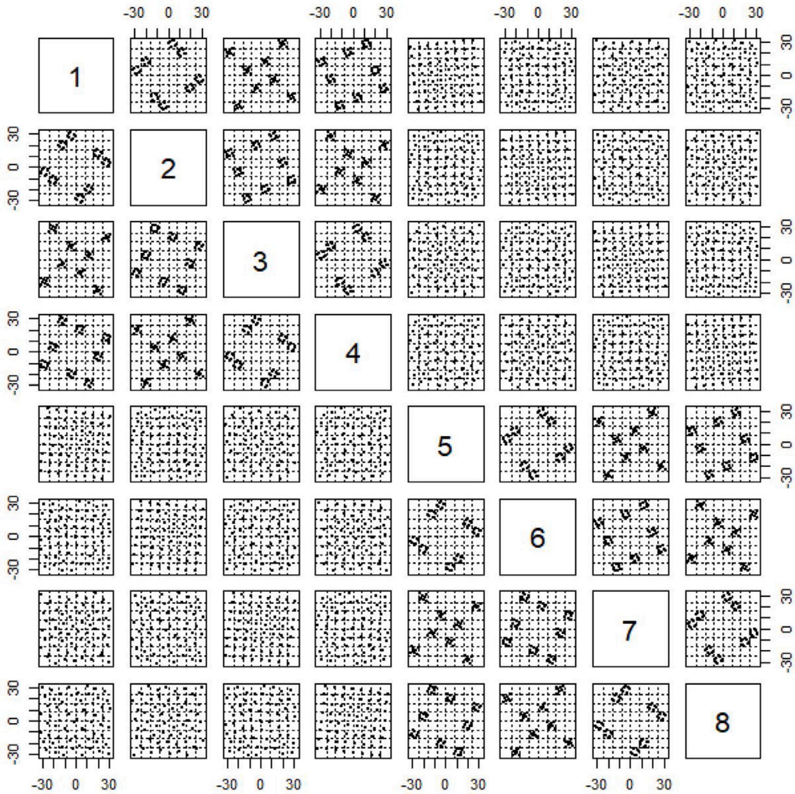

Fig. 1. The two-dimensional projection of the first 8 columns of the $OLHD(64, 57)$.

The following example is illustrated the stratifications in Theorem 4.

Example 4. Continue with Example 1, we give the stratifications on two-dimensional projection of the $OLHD(64, 57)$. Here $u = 3, v = 0$. Any two distinct columns achieve a stratification on a 2×2 grid; any three distinct columns from the same group Z_i^* achieve a stratification on a $2 \times 2 \times 2$ grid. For the stratifications on 2×4, 4×2 or 4×4 grids, to save space, we only show the result of the first 8 columns which are in the same group of the $OLHD(64, 57)$ (see Fig. 1 and Table 2).

According to our simulation, different groups also have good stratification results. Generally, the number of the column pairs achieving the stratification on low-dimension projections between different groups are more than that from the same group. However it is difficult to determine the accurate number, we only give an example to illustrate it, further theoretical researches are not discussed in this paper.

Example 5. Now, we discuss the stratifications about column pairs from different groups in $OLHD(128, 113)$. We only give the stratification results of two groups since other cases are similar. Take the first group and the second group as an example. Note that each group has 16 columns. There are 256 column pairs achieving a stratification on a 2×4 grid, 192 column pairs on a 4×4 grid, 128 column pairs on a 4×8 grid or a 8×8 grid. To save space, we just show the figure about column 1 in the first group and columns 21–24, 29–32 in second group. The first row or the first column of the following two pictures characterize the stratification of first column and the 21–24th columns and 29–32th columns on two-dimension projection, which achieve stratifications on 8×8 grids (see Fig. 2).

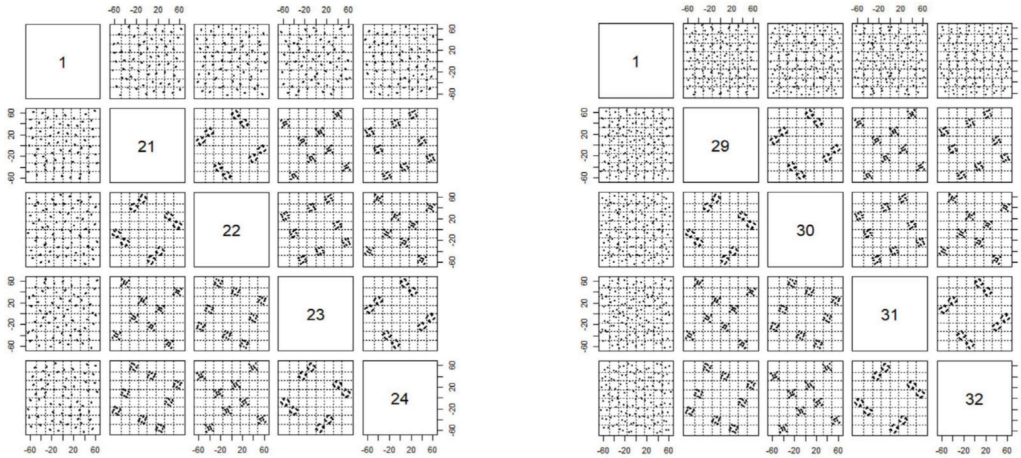

Fig. 2. The two-dimensional projection of the 1st column and the 21–24th, 29–32th columns of the $OLHD(128, 113)$.

6. Discussion

In this paper, we initially generate two types of rotation matrices. Utilizing the characteristics of them and grouping methods, we introduce some novel OLHDs. The number of the columns in the proposed OLHDs can either match or surpass those presented by Sun and Tang (2017).

The proposed design is obtained by grouping the orthogonal arrays and then rotating them. This construction method ensures that the obtained OLHD not only has orthogonality, but also has good low-dimensional projection uniformity. The key to this method is the construction of the rotation matrix and the grouping of the corresponding columns of the orthogonal arrays. In the future, we will continue to explore and study the construction of rotation matrices with flexible orders, and the grouping method of the corresponding columns of the orthogonal arrays. We hope that more OLHDs with good space-filling properties and flexible parameters can be constructed.

Acknowledgments

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Appendix

A.1. Proof of Lemma 2

First, we prove that S_{u0} is a rotation matrix. For $u = 2$, $S_{20}^T S_{20} = c_2 I_4$, where $c_2 = p^4 + p^2 + 1$. Assume $S_{(k-1)0}^T S_{(k-1)0} = c_{k-1} I_{2^{k-1}}$ is true for $u = k - 1 (k \geq 3)$. For $u = k$, we have

$$S_{k0}^T S_{k0} = \begin{bmatrix} (p^{3\cdot 2^{k-2}} + 1) S_{(k-1)0}^T S_{(k-1)0} & 0 \\ 0 & (p^{3\cdot 2^{k-2}} + 1) S_{(k-1)0}^T S_{(k-1)0} \end{bmatrix},$$

the result is true by induction hypothesis.

Next, we prove S_{uv} is the rotation matrix. Since

$$S_{uv}^T S_{uv} = \begin{bmatrix} p^2 S_{u(v-1)}^T S_{u(v-1)} + Q_{u+v-1}^T Q_{u+v-1} & -p^2 S_{u(v-1)}^T Q_{u+v-1} + p Q_{u+v-1}^T S_{u(v-1)} \\ -p Q_{u+v-1}^T S_{u(v-1)} + p S_{u(v-1)}^T Q_{u+v-1} & Q_{u+v-1}^T Q_{u+v-1} + p^2 S_{u(v-1)}^T S_{u(v-1)} \end{bmatrix},$$

the result can be obtained if $S_{u(v-1)}^T Q_{u+v-1} = Q_{u+v-1}^T S_{u(v-1)}$ holds. By a simple induction argument, it is obvious.

Table 3
The OLHD(64,57) constructed in Example 5.

number	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
1	−31.5	−4.5	−22.5	−13.5	−24.5	−3.5	−17.5	−10.5	−31.5	−4.5	−22.5	−13.5	−24.5	−3.5	−17.5	−10.5	−31.5	−4.5	−22.5	−13.5	−24.5	−3.5	−17.5	−10.5	−31.5	−4.5	−22.5	−13.5
2	−30.5	−5.5	−16.5	−11.5	−16.5	−11.5	30.5	5.5	−28.5	−0.5	−18.5	−14.5	−0.5	28.5	14.5	−18.5	−26.5	−6.5	−19.5	−15.5	15.5	−19.5	6.5	−26.5	−16.5	−11.5	30.5	5.5
3	−29.5	−1.5	−20.5	−8.5	−8.5	20.5	−1.5	29.5	−25.5	−2.5	−23.5	−12.5	23.5	12.5	−25.5	−2.5	−21.5	−9.5	27.5	7.5	−9.5	21.5	−7.5	27.5	−1.5	29.5	8.5	−20.5
4	−28.5	−0.5	−18.5	−14.5	−0.5	28.5	14.5	−18.5	−26.5	−6.5	−19.5	−15.5	15.5	−19.5	6.5	−26.5	−16.5	−11.5	30.5	5.5	30.5	5.5	16.5	11.5	−14.5	18.5	−0.5	28.5
5	−27.5	−7.5	−21.5	−9.5	7.5	−27.5	−9.5	21.5	−19.5	−15.5	26.5	6.5	6.5	−26.5	−15.5	19.5	−11.5	16.5	−5.5	30.5	5.5	−30.5	−11.5	16.5	29.5	1.5	20.5	8.5
6	−26.5	−6.5	−19.5	−15.5	15.5	−19.5	6.5	−26.5	−16.5	−11.5	30.5	5.5	30.5	5.5	16.5	11.5	−14.5	18.5	−0.5	28.5	−18.5	−14.5	28.5	0.5	18.5	14.5	−28.5	−0.5
7	−25.5	−2.5	−23.5	−12.5	23.5	12.5	−25.5	−2.5	−21.5	−9.5	27.5	7.5	−9.5	21.5	−7.5	27.5	−1.5	29.5	8.5	−20.5	20.5	8.5	−29.5	−1.5	3.5	−24.5	−10.5	17.5
8	−24.5	−3.5	−17.5	−10.5	31.5	4.5	22.5	13.5	−22.5	−13.5	31.5	4.5	−17.5	−10.5	24.5	3.5	−4.5	31.5	13.5	−22.5	−3.5	24.5	10.5	−17.5	12.5	−23.5	2.5	−25.5
9	−23.5	−12.5	25.5	2.5	−25.5	−2.5	−23.5	−12.5	−7.5	27.5	9.5	−21.5	−27.5	−7.5	−21.5	−9.5	9.5	−21.5	7.5	−27.5	−21.5	−9.5	27.5	7.5	26.5	6.5	19.5	15.5
10	−22.5	−13.5	31.5	4.5	−17.5	−10.5	24.5	3.5	−4.5	31.5	13.5	−22.5	−3.5	24.5	10.5	−17.5	12.5	−23.5	2.5	−25.5	2.5	−25.5	−12.5	23.5	21.5	9.5	−27.5	−7.5
11	−21.5	−9.5	27.5	7.5	−9.5	21.5	−7.5	27.5	−1.5	29.5	8.5	−20.5	20.5	8.5	−29.5	−1.5	3.5	−24.5	−10.5	17.5	−4.5	31.5	13.5	−22.5	4.5	−31.5	−13.5	22.5
12	−20.5	−8.5	29.5	1.5	−1.5	29.5	8.5	−20.5	−2.5	25.5	12.5	−23.5	12.5	−23.5	2.5	−25.5	6.5	−26.5	−15.5	19.5	19.5	15.5	−26.5	−6.5	11.5	−16.5	5.5	−30.5
13	−19.5	−15.5	26.5	6.5	6.5	−26.5	−15.5	19.5	−11.5	16.5	−5.5	30.5	5.5	−30.5	−11.5	16.5	29.5	1.5	20.5	8.5	8.5	−20.5	1.5	−29.5	−24.5	−3.5	−17.5	−10.5
14	−18.5	−14.5	28.5	0.5	14.5	−18.5	0.5	−28.5	−8.5	20.5	−1.5	29.5	29.5	1.5	20.5	8.5	24.5	3.5	17.5	10.5	−31.5	−4.5	−22.5	−13.5	−23.5	−12.5	25.5	2.5
15	−17.5	−10.5	24.5	3.5	22.5	13.5	−31.5	−4.5	−13.5	22.5	−4.5	31.5	−10.5	17.5	−3.5	24.5	23.5	12.5	−25.5	−2.5	25.5	2.5	23.5	12.5	−6.5	26.5	15.5	−19.5
16	−16.5	−11.5	30.5	5.5	30.5	5.5	16.5	11.5	−14.5	18.5	−0.5	28.5	−18.5	−14.5	28.5	0.5	18.5	14.5	−28.5	−0.5	−14.5	18.5	−0.5	28.5	−9.5	21.5	−7.5	27.5
17	−15.5	19.5	−6.5	26.5	−26.5	−6.5	−19.5	−15.5	17.5	10.5	−24.5	−3.5	−22.5	−13.5	31.5	4.5	−12.5	23.5	−2.5	25.5	−2.5	25.5	12.5	−23.5	20.5	8.5	−29.5	−1.5
18	−14.5	18.5	−0.5	28.5	−18.5	−14.5	28.5	0.5	18.5	14.5	−28.5	−0.5	−14.5	18.5	−0.5	28.5	−9.5	21.5	−7.5	27.5	21.5	9.5	−27.5	−7.5	27.5	7.5	21.5	9.5
19	−13.5	22.5	−4.5	31.5	−10.5	17.5	−3.5	24.5	23.5	12.5	−25.5	−2.5	25.5	2.5	23.5	12.5	−6.5	26.5	15.5	−19.5	−19.5	−15.5	26.5	6.5	10.5	−17.5	3.5	−24.5
20	−12.5	23.5	−2.5	25.5	−2.5	25.5	12.5	−23.5	20.5	8.5	−29.5	−1.5	1.5	−29.5	−8.5	20.5	−3.5	24.5	10.5	−17.5	4.5	−31.5	−13.5	22.5	5.5	−30.5	−11.5	16.5
21	−11.5	16.5	−5.5	30.5	5.5	−30.5	−11.5	16.5	29.5	1.5	20.5	8.5	8.5	−20.5	1.5	−29.5	−24.5	−3.5	−17.5	−10.5	31.5	4.5	22.5	13.5	−22.5	−13.5	31.5	4.5
22	−10.5	17.5	−3.5	24.5	13.5	−22.5	4.5	−31.5	30.5	5.5	16.5	11.5	16.5	11.5	−30.5	−5.5	−29.5	−1.5	−20.5	−8.5	−8.5	20.5	−1.5	29.5	−25.5	−2.5	−23.5	−12.5
23	−9.5	21.5	−7.5	27.5	21.5	9.5	−27.5	−7.5	27.5	7.5	21.5	9.5	−7.5	27.5	9.5	−21.5	−18.5	−14.5	28.5	0.5	14.5	−18.5	0.5	−28.5	−8.5	20.5	−1.5	29.5
24	−8.5	20.5	−1.5	29.5	29.5	1.5	20.5	8.5	24.5	3.5	17.5	10.5	−31.5	−4.5	−22.5	−13.5	−23.5	−12.5	25.5	2.5	−25.5	−2.5	−23.5	−12.5	−7.5	27.5	9.5	−21.5
25	−7.5	27.5	9.5	−21.5	−27.5	−7.5	−21.5	−9.5	9.5	−21.5	7.5	−27.5	−21.5	−9.5	27.5	7.5	26.5	6.5	19.5	15.5	−15.5	19.5	−6.5	26.5	−17.5	−10.5	24.5	3.5
26	−6.5	26.5	15.5	−19.5	−19.5	−15.5	26.5	6.5	10.5	−17.5	3.5	−24.5	−13.5	22.5	−4.5	31.5	31.5	4.5	22.5	13.5	24.5	3.5	17.5	10.5	−30.5	−5.5	−16.5	−11.5
27	−5.5	30.5	11.5	−16.5	−11.5	16.5	−5.5	30.5	15.5	−19.5	6.5	−26.5	26.5	6.5	19.5	15.5	16.5	11.5	−30.5	−5.5	−30.5	−5.5	−16.5	−11.5	−15.5	19.5	−6.5	26.5
28	−4.5	31.5	13.5	−22.5	−3.5	24.5	10.5	−17.5	12.5	−23.5	2.5	−25.5	2.5	−25.5	−12.5	23.5	21.5	9.5	−27.5	−7.5	9.5	−21.5	7.5	−27.5	−0.5	28.5	14.5	−18.5
29	−3.5	24.5	10.5	−17.5	4.5	−31.5	−13.5	22.5	5.5	−30.5	−11.5	16.5	11.5	−16.5	5.5	−30.5	14.5	−18.5	0.5	−28.5	18.5	14.5	−28.5	−0.5	19.5	15.5	−26.5	−6.5
30	−2.5	25.5	12.5	−23.5	12.5	−23.5	2.5	−25.5	6.5	−26.5	−15.5	19.5	19.5	15.5	−26.5	−6.5	11.5	−16.5	5.5	−30.5	−5.5	30.5	11.5	−16.5	28.5	0.5	18.5	14.5
31	−1.5	29.5	8.5	−20.5	20.5	8.5	−29.5	−1.5	3.5	−24.5	−10.5	17.5	−4.5	31.5	13.5	−22.5	4.5	−31.5	−13.5	22.5	3.5	−24.5	−10.5	17.5	13.5	−22.5	4.5	−31.5
32	−0.5	28.5	14.5	−18.5	28.5	0.5	18.5	14.5	0.5	−28.5	−14.5	18.5	−28.5	−0.5	−18.5	−14.5	1.5	−29.5	−8.5	20.5	−20.5	−8.5	29.5	1.5	2.5	−25.5	−12.5	23.5

number	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57
1	−24.5	−3.5	−17.5	−10.5	−31.5	−4.5	−22.5	−13.5	−24.5	−3.5	−17.5	−10.5	−31.5	−4.5	−22.5	−13.5	−24.5	−3.5	−17.5	−10.5	−31.5	−4.5	−22.5	−13.5	−24.5	−3.5	−17.5	−10.5	−31.5
2	30.5	5.5	16.5	11.5	−14.5	18.5	−0.5	28.5	−18.5	−14.5	28.5	0.5	26.5	6.5	19.5	15.5	−15.5	19.5	−6.5	26.5	−8.5	20.5	−1.5	29.5	29.5	1.5	20.5	8.5	24.5
3	20.5	8.5	−29.5	−1.5	11.5	−16.5	5.5	−30.5	−5.5	30.5	11.5	−16.5	13.5	−31.5	22.5	4.5	10.5	−24.5	17.5	3.5	−6.5	26.5	15.5	−19.5	−19.5	−15.5	26.5	6.5	31.5
4	−18.5	−14.5	28.5	0.5	26.5	6.5	19.5	15.5	−15.5	19.5	−6.5	26.5	−8.5	29.5	−19.5	−6.5	29.5	8.5	6.5	−19.5	−17.5	−10.5	24.5	3.5	22.5	13.5	−31.5	−4.5	−24.5
5	8.5	−20.5	1.5	−29.5	−25.5	−2.5	−23.5	−12.5	23.5	12.5	−25.5	−2.5	−4.5	31.5	13.5	−22.5	−3.5	24.5	10.5	−17.5	7.5	−27.5	−9.5	21.5	27.5	7.5	21.5	9.5	−2.5
6	−14.5	18.5	−0.5	28.5	−8.5	20.5	−1.5	29.5	29.5	1.5	20.5	8.5	1.5	−29.5	−8.5	20.5	−20.5	−8.5	29.5	1.5	16.5	11.5	−30.5	−5.5	−30.5	−5.5	−16.5	−11.5	5.5
7	−4.5	31.5	13.5	−22.5	13.5	−22.5	4.5	−31.5	10.5	−17.5	3.5	−24.5	22.5	4.5	−13.5	31.5	17.5	3.5	−10.5	24.5	30.5	5.5	16.5	11.5	16.5	11.5	−30.5	−5.5	2.5
8	2.5	−25.5	−12.5	23.5	28.5	0.5	18.5	14.5	0.5	−28.5	−14.5	18.5	−19.5	−6.5	8.5	−29.5	6.5	−19.5	−29.5	−8.5	9.5	−21.5	7.5	−27.5	−21.5	−9.5	27.5	7.5	−5.5
9	−15.5	19.5	−6.5	26.5	−16.5	−11.5	30.5	5.5	30.5	5.5	16.5	11.5	−13.5	23.5	−6.5	27.5	−10.5	25.5	−19.5	−7.5	25.5	2.5	23.5	12.5	−23.5	−12.5	25.5	2.5	−17.5
10	9.5	−21.5	7.5	−27.5	−1.5	29.5	8.5	−20.5	20.5	8.5	−29.5	−1.5	8.5	−21.5	3.5	−25.5	−29.5	−9.5	−4.5	23.5	14.5	−18.5	0.5	−28.5	18.5	14.5	−28.5	−0.5	22.5
11	3.5	−24.5	−10.5	17.5	4.5	−31.5	−13.5	22.5	3.5	−24.5	−10.5	17.5	31.5	12.5	6.5	−18.5	24.5	2.5	19.5	14.5	0.5	−28.5	−14.5	18.5	−28.5	−0.5	−18.5	−14.5	17.5
12	−5.5	30.5	11.5	−16.5	21.5	9.5	−27.5	−7.5	9.5	−21.5	7.5	−27.5	−26.5	−14.5	−3.5	16.5	15.5	−18.5	4.5	−30.5	23.5	12.5	−25.5	−2.5	25.5	2.5	23.5	12.5	−22.5
13	31.5	4.5	22.5	13.5	−22.5	−13.5	31.5	4.5	−17.5	−10.5	24.5	3.5	−22.5	−12.5	29.5	0.5	−17.5	−2.5	8.5	−28.5	−1.5	29.5	8.5	−20.5	20.5	8.5	−29.5	−1.5	−12.5
14	−25.5	−2.5	−23.5	−12.5	−7.5	27.5	9.5	−21.5	−27.5	−7.5	−21.5	−9.5	19.5	14.5	−24.5	−2.5	−6.5	18.5	31.5	12.5	−22.5	−13.5	31.5	4.5	−17.5	−10.5	24.5	3.5	11.5
15	−19.5	−15.5	26.5	6.5	2.5	−25.5	−12.5	23.5	−12.5	23.5	−2.5	25.5	4.5	−23.5	−29.5	−9.5	3.5	−25.5	−8.5	21.5	−24.5	−3.5	−17.5	−10.5	31.5	4.5	22.5	13.5	12.5
16	21.5	9.5	−27.5	−7.5	19.5	15.5	−26.5	−6.5	−6.5	26.5	15.5	−19.5	−1.5	21.5	24.5	11.5	20.5	9.5	−31.5	−5.5	−15.5	19.5	−6.5	26.5	−26.5	−6.5	−19.5	−15.5	−11.5
17	1.5	−29.5	−8.5	20.5	−2.5	25.5	12.5	−23.5	12.5	−23.5	2.5	−25.5	14.5	−27.5	18.5	7.5	18.5	7.5	−14.5	27.5	−16.5	−11.5	30.5	5.5	30.5	5.5	16.5	11.5	−26.5
18	−7.5	27.5	9.5	−21.5	−19.5	−15.5	26.5	6.5	6.5	−26.5	−15.5	19.5	−11.5	25.5	−23.5	−5.5	5.5	−23.5	−25.5	−11.5	−7.5	27.5	9.5	−21.5	−27.5	−7.5	−21.5	−9.5	29.5
19	−13.5	22.5	−4.5	31.5	22.5	13.5	−31.5	−4.5	17.5	10.5	−24.5	−3.5	−28.5	−0.5	−18.5	−14.5	−0.5	28.5	14.5	−18.5	−9.5	21.5	−7.5	27.5	21.5	9.5	−27.5	−7.5	26.5
20	11.5	−16.5	5.5	−30.5	7.5	−27.5	−9.5	21.5	27.5	7.5	21.5	9.5	25.5	2.5	23.5	12.5	−23.5	−12.5	25.5	2.5	−30.5	−5.5	−16.5	−11.5	−16.5	−11.5	30.5	5.5	−29.5
21	−17.5	−10.5	24.5	3.5	−4.5	31.5	13.5	−22.5	−3.5	24.5	10.5	−17.5	21.5	0.5	−9.5	28.5	9.5	−28.5	21.5	0.5	8.5	−20.5	1.5	−29.5	−29.5	−1.5	−20.5	−8.5	−7.5
22	23.5	12.5	−25.5	−2.5	−21.5	−9.5	27.5	7.5	−9.5	21.5	−7.5	27.5	−16.5	−2.5	12.5	−30.5	30.5	12.5	2.5	−16.5	31.5	4.5	22.5	13.5	24.5	3.5	17.5	10.5	0.5
23	29.5	1.5	20.5	8.5	16.5	11.5	−30.5	−5.5	−30.5	−5.5	−16.5	−11.5	−7.5	27.5	9.5	−21.5	−27.5	−7.5	−21.5	−9.5	17.5	10.5	−24.5	−3.5	−22.5	−13.5	31.5	4.5	7.5
24	−27.5	−7.5	−21.5	−9.5	1.5	−29.5	−8.5	20.5	−20.5	−8.5	29.5	1.5	2.5	−25.5	−12.5	23.5	−12.5	23.5	−2.5	25.5	6.5	−26.5	−15.5	19.5	19.5	15.5	−26.5	−6.5	−0.5
25	22.5	13.5	−31.5	−4.5	−13.5	22.5	−4.5	31.5	−10.5	17.5	−3.5	24.5	28.5	8.5	2.5	−17.5	0.5	−29.5	−12.5	22.5	22.5	13.5	−31.5	−4.5	17.5	10.5	−24.5	−3.5	−20.5
26	−16.5	−11.5	30.5	5.5	−28.5	−0.5	−18.5	−14.5	−0.5	28.5	14.5	−18.5	−25.5	−10.5	−7.5	19.5	23.5	13.5	−27.5	−6.5	1.5	−29.5	−8.5	20.5	−20.5	−8.5	29.5	1.5	19.5
27	−26.5	−6.5	−19.5	−15.5	25.5	2.5	23.5	12.5	−23.5	−12.5	25.5	2.5	−14.5	19.5	−2.5	24.5	−18.5	−6.5	12.5	−31.5	15.5	−19.5	6.5	−26.5	26.5	6.5	19.5	15.5	20.5
28	28.5	0.5	18.5	14.5	8.5	−20.5	1.5	−29.5	−29.5	−1.5	−20.5	−8.5	11.5	−17.5	7.5	−26.5	−5.5	22.5	27.5	15.5	24.5	3.5	17.5	10.5	−31.5	−4.5	−22.5	−13.5	−19.5
29	−6.5	26.5	15.5	−19.5	−11.5	16.5	−5.5	30.5	5.5	−30.5	−11.5	16.5	7.5	−19.5	−25.5	−10.5	27.5	6.5	23.5	13.5	−14.5	18.5	−0.5	28.5	−18.5	−14.5	28.5	0.5	−9.5
30	0.5	−28.5	−14.5	18.5	−26.5	−6.5	−19.5	−15.5	15.5	−19.5	6.5	−26.5	−2.5	17.5	28.5	8.5	12.5	−22.5	0.5	−29.5	−25.5	−2.5	−23.5	−12.5	23.5	12.5	−25.5	−2.5	14.5
31	10.5	−17.5	3.5	−24.5	31.5	4.5	22.5	13.5	24.5	3.5	17.5	10.5	−21.5	−8.5	25.5	3.5	−9.5	29.5	−23.5	−4.5	−23.5	−12.5	25.5	2.5	−25.5	−2.5	−23.5	−12.5	9.5
32	−12.5	23.5	−2.5	25.5	14.5	−18.5	0.5	−28.5	18.5	14.5	−28.5	−0.5	16.5	10.5	−28.5	−1.5	−30.5	−13.5	−0.5	20.5	−0.5	28.5	14.5	−18.5	28.5	0.5	18.5	14.5	−14.5

number	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
33	0.5	−28.5	−14.5	18.5	−28.5	−0.5	−18.5	−14.5	1.5	−29.5	−8.5	20.5	−20.5	−8.5	29.5	1.5	2.5	−25.5	−12.5	23.5	−12.5	23.5	−2.5	25.5	7.5	−27.5	−9.5	21.5
34	1.5	−29.5	−8.5	20.5	−20.5	−8.5	29.5	1.5	2.5	−25.5	−12.5	23.5	−12.5	23.5	−2.5	25.5	7.5	−27.5	−9.5	21.5	27.5	7.5	21.5	9.5	8.5	−20.5	1.5	−29.5
35	2.5	−25.5	−12.5	23.5	−12.5	23.5	−2.5	25.5	7.5	−27.5	−9.5	21.5	27.5	7.5	21.5	9.5	8.5	−20.5	1.5	−29.5	−29.5	−1.5	−20.5	−8.5	25.5	2.5	23.5	12.5
36	3.5	−24.5	−10.5	17.5	−4.5	31.5	13.5	−22.5	4.5	−31.5	−13.5	22.5	3.5	−24.5	−10.5	17.5	13.5	−22.5	4.5	−31.5	10.5	−17.5	3.5	−24.5	22.5	13.5	−31.5	−4.5
37	4.5	−31.5	−13.5	22.5	3.5	−24.5	−10.5	17.5	13.5	−22.5	4.5	−31.5	10.5	−17.5	3.5	−24.5	22.5	13.5	−31.5	−4.5	17.5	10.5	−24.5	−3.5	−5.5	30.5	11.5	−16.5
38	5.5	−30.5	−11.5	16.5	11.5	−16.5	5.5	−30.5	14.5	−18.5	0.5	−28.5	18.5	14.5	−28.5	−0.5	19.5	15.5	−26.5	−6.5	−6.5	26.5	15.5	−19.5	−10.5	17.5	−3.5	24.5
39	6.5	−26.5	−15.5	19.5	19.5	15.5	−26.5	−6.5	11.5	−16.5	5.5	−30.5	−5.5	30.5	11.5	−16.5	28.5	0.5	18.5	14.5	0.5	−28.5	−14.5	18.5	−27.5	−7.5	−21.5	−9.5
40	7.5	−27.5	−9.5	21.5	27.5	7.5	21.5	9.5	8.5	−20.5	1.5	−29.5	−29.5	−1.5	−20.5	−8.5	25.5	2.5	23.5	12.5	−23.5	−12.5	25.5	2.5	−20.5	−8.5	29.5	1.5
41	8.5	−20.5	1.5	−29.5	−29.5	−1.5	−20.5	−8.5	25.5	2.5	23.5	12.5	−23.5	−12.5	25.5	2.5	−20.5	−8.5	29.5	1.5	−1.5	29.5	8.5	−20.5	−2.5	25.5	12.5	−23.5
42	9.5	−21.5	7.5	−27.5	−21.5	−9.5	27.5	7.5	26.5	6.5	19.5	15.5	−15.5	19.5	−6.5	26.5	−17.5	−10.5	24.5	3.5	22.5	13.5	−31.5	−4.5	−13.5	22.5	−4.5	31.5

(continued on next page)

Table 3 (*continued*).

43	10.5	−17.5	3.5	−24.5	−13.5	22.5	−4.5	31.5	31.5	4.5	22.5	13.5	24.5	3.5	17.5	10.5	−30.5	−5.5	−16.5	−11.5	−16.5	−11.5	30.5	5.5	−28.5	−0.5	−18.5	−14.5
44	11.5	−16.5	5.5	−30.5	−5.5	30.5	11.5	−16.5	28.5	0.5	18.5	14.5	0.5	−28.5	−14.5	18.5	−27.5	−7.5	−21.5	−9.5	7.5	−27.5	−9.5	21.5	−19.5	−15.5	26.5	6.5
45	12.5	−23.5	2.5	−25.5	2.5	−25.5	−12.5	23.5	21.5	9.5	−27.5	−7.5	9.5	−21.5	7.5	−27.5	−0.5	28.5	14.5	−18.5	28.5	0.5	18.5	14.5	0.5	−28.5	−14.5	18.5
46	13.5	−22.5	4.5	−31.5	10.5	−17.5	3.5	−24.5	22.5	13.5	−31.5	−4.5	17.5	10.5	−24.5	−3.5	−5.5	30.5	11.5	−16.5	−11.5	16.5	−5.5	30.5	15.5	−19.5	6.5	−26.5
47	14.5	−18.5	0.5	−28.5	18.5	14.5	−28.5	−0.5	19.5	15.5	−26.5	−6.5	−6.5	26.5	15.5	−19.5	−10.5	17.5	−3.5	24.5	13.5	−22.5	4.5	−31.5	30.5	5.5	16.5	11.5
48	15.5	−19.5	6.5	−26.5	26.5	6.5	19.5	15.5	16.5	11.5	−30.5	−5.5	−30.5	−5.5	−16.5	−11.5	−15.5	19.5	−6.5	26.5	−26.5	−6.5	−19.5	−15.5	17.5	10.5	−24.5	−3.5
49	16.5	11.5	−30.5	−5.5	−30.5	−5.5	−16.5	−11.5	−15.5	19.5	−6.5	26.5	−26.5	−6.5	−19.5	−15.5	17.5	10.5	−24.5	−3.5	−22.5	−13.5	31.5	4.5	−12.5	23.5	−2.5	25.5
50	17.5	10.5	−24.5	−3.5	−22.5	−13.5	31.5	4.5	−12.5	23.5	−2.5	25.5	−2.5	25.5	12.5	−23.5	20.5	8.5	−29.5	−1.5	1.5	−29.5	−8.5	20.5	−3.5	24.5	10.5	−17.5
51	18.5	14.5	−28.5	−0.5	−14.5	18.5	−0.5	28.5	−9.5	21.5	−7.5	27.5	21.5	9.5	−27.5	−7.5	27.5	7.5	21.5	9.5	−7.5	27.5	9.5	−21.5	−18.5	−14.5	28.5	0.5
52	19.5	15.5	−26.5	−6.5	−6.5	26.5	15.5	−19.5	−10.5	17.5	−3.5	24.5	13.5	−22.5	4.5	−31.5	30.5	5.5	16.5	11.5	16.5	11.5	−30.5	−5.5	−29.5	−1.5	−20.5	−8.5
53	20.5	8.5	−29.5	−1.5	1.5	−29.5	−8.5	20.5	−3.5	24.5	10.5	−17.5	4.5	−31.5	−13.5	22.5	5.5	−30.5	−11.5	16.5	11.5	−16.5	5.5	−30.5	14.5	−18.5	0.5	−28.5
54	21.5	9.5	−27.5	−7.5	9.5	−21.5	7.5	−27.5	−0.5	28.5	14.5	−18.5	28.5	0.5	18.5	14.5	0.5	−28.5	−14.5	18.5	−28.5	−0.5	−18.5	−14.5	1.5	−29.5	−8.5	20.5
55	22.5	13.5	−31.5	−4.5	17.5	10.5	−24.5	−3.5	−5.5	30.5	11.5	−16.5	−11.5	16.5	−5.5	30.5	15.5	−19.5	6.5	−26.5	26.5	6.5	19.5	15.5	16.5	11.5	−30.5	−5.5
56	23.5	12.5	−25.5	−2.5	25.5	2.5	23.5	12.5	−6.5	26.5	15.5	−19.5	−19.5	−15.5	26.5	6.5	10.5	−17.5	3.5	−24.5	−13.5	22.5	−4.5	31.5	31.5	4.5	22.5	13.5
57	24.5	3.5	17.5	10.5	−31.5	−4.5	−22.5	−13.5	−23.5	−12.5	25.5	2.5	−25.5	−2.5	−23.5	−12.5	−7.5	27.5	9.5	−21.5	−27.5	−7.5	−21.5	−9.5	9.5	−21.5	7.5	−27.5
58	25.5	2.5	23.5	12.5	−23.5	−12.5	25.5	2.5	−20.5	−8.5	29.5	1.5	−1.5	29.5	8.5	−20.5	−2.5	25.5	12.5	−23.5	12.5	−23.5	2.5	−25.5	6.5	−26.5	−15.5	19.5
59	26.5	6.5	19.5	15.5	−15.5	19.5	−6.5	26.5	−17.5	−10.5	24.5	3.5	22.5	13.5	−31.5	−4.5	−13.5	22.5	−4.5	31.5	−10.5	17.5	−3.5	24.5	23.5	12.5	−25.5	−2.5
60	27.5	7.5	21.5	9.5	−7.5	27.5	9.5	−21.5	−18.5	−14.5	28.5	0.5	14.5	−18.5	0.5	−28.5	−8.5	20.5	−1.5	29.5	29.5	1.5	20.5	8.5	24.5	3.5	17.5	10.5
61	28.5	0.5	18.5	14.5	0.5	−28.5	−14.5	18.5	−27.5	−7.5	−21.5	−9.5	7.5	−27.5	−9.5	21.5	−19.5	−15.5	26.5	6.5	6.5	−26.5	−15.5	19.5	−11.5	16.5	−5.5	30.5
62	29.5	1.5	20.5	8.5	8.5	−20.5	1.5	−29.5	−24.5	−3.5	−17.5	−10.5	31.5	4.5	22.5	13.5	−22.5	−13.5	31.5	4.5	−17.5	−10.5	24.5	3.5	−4.5	31.5	13.5	−22.5
63	30.5	5.5	16.5	11.5	16.5	11.5	−30.5	−5.5	−29.5	−1.5	−20.5	−8.5	−8.5	20.5	−1.5	29.5	−25.5	−2.5	−23.5	−12.5	23.5	12.5	−25.5	−2.5	−21.5	−9.5	27.5	7.5
64	31.5	4.5	22.5	13.5	24.5	3.5	17.5	10.5	−30.5	−5.5	−16.5	−11.5	−16.5	−11.5	30.5	5.5	−28.5	−0.5	−18.5	−14.5	−0.5	28.5	14.5	−18.5	−26.5	−6.5	−19.5	−15.5

number	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57
33	27.5	7.5	21.5	9.5	0.5	−28.5	−14.5	18.5	−28.5	−0.5	−18.5	−14.5	3.5	−24.5	−10.5	17.5	−4.5	31.5	13.5	−22.5	12.5	−23.5	2.5	−25.5	2.5	−25.5	−12.5	23.5	30.5
34	−29.5	−1.5	−20.5	−8.5	17.5	10.5	−24.5	−3.5	−22.5	−13.5	31.5	4.5	−6.5	26.5	15.5	−19.5	−19.5	−15.5	26.5	6.5	27.5	7.5	21.5	9.5	−7.5	27.5	9.5	−21.5	−25.5
35	−23.5	−12.5	25.5	2.5	−20.5	−8.5	29.5	1.5	−1.5	29.5	8.5	−20.5	−17.5	−3.5	10.5	−24.5	22.5	4.5	−13.5	31.5	21.5	9.5	−27.5	−7.5	9.5	−21.5	7.5	−27.5	−30.5
36	17.5	10.5	−24.5	−3.5	−5.5	30.5	11.5	−16.5	−11.5	16.5	−5.5	30.5	20.5	1.5	−15.5	26.5	1.5	−20.5	−26.5	−15.5	2.5	−25.5	−12.5	23.5	−12.5	23.5	−2.5	25.5	25.5
37	−11.5	16.5	−5.5	30.5	6.5	−26.5	−15.5	19.5	19.5	15.5	−26.5	−6.5	24.5	3.5	17.5	10.5	−31.5	−4.5	−22.5	−13.5	−20.5	−8.5	29.5	1.5	−1.5	29.5	8.5	−20.5	3.5
38	13.5	−22.5	4.5	−31.5	23.5	12.5	−25.5	−2.5	25.5	2.5	23.5	12.5	−29.5	−1.5	−20.5	−8.5	−8.5	20.5	−1.5	29.5	−3.5	24.5	10.5	−17.5	4.5	−31.5	−13.5	22.5	−4.5
39	7.5	−27.5	−9.5	21.5	−18.5	−14.5	28.5	0.5	14.5	−18.5	0.5	−28.5	−10.5	24.5	−17.5	−3.5	13.5	−31.5	22.5	4.5	−13.5	22.5	−4.5	31.5	−10.5	17.5	−3.5	24.5	−3.5
40	−1.5	29.5	8.5	−20.5	−3.5	24.5	10.5	−17.5	4.5	−31.5	−13.5	22.5	15.5	−26.5	20.5	1.5	26.5	15.5	1.5	−20.5	−26.5	−6.5	−19.5	−15.5	15.5	−19.5	6.5	−26.5	4.5
41	12.5	−23.5	2.5	−25.5	15.5	−19.5	6.5	−26.5	26.5	6.5	19.5	15.5	17.5	11.5	−26.5	−7.5	−22.5	−5.5	15.5	−27.5	−10.5	17.5	−3.5	24.5	13.5	−22.5	4.5	−31.5	16.5
42	−10.5	17.5	−3.5	24.5	30.5	5.5	16.5	11.5	16.5	11.5	−30.5	−5.5	−20.5	−9.5	31.5	5.5	−1.5	21.5	24.5	11.5	−29.5	−1.5	−20.5	−8.5	−8.5	20.5	−1.5	29.5	−23.5
43	−0.5	28.5	14.5	−18.5	−27.5	−7.5	−21.5	−9.5	7.5	−27.5	−9.5	21.5	−3.5	16.5	26.5	14.5	4.5	−30.5	−15.5	18.5	−19.5	−15.5	26.5	6.5	6.5	−26.5	−15.5	19.5	−16.5
44	6.5	−26.5	−15.5	19.5	−10.5	17.5	−3.5	24.5	13.5	−22.5	4.5	−31.5	6.5	−18.5	−31.5	−12.5	19.5	14.5	−24.5	−2.5	−4.5	31.5	13.5	−22.5	−3.5	24.5	10.5	−17.5	23.5
45	−28.5	−0.5	−18.5	−14.5	9.5	−21.5	7.5	−27.5	−21.5	−9.5	27.5	7.5	10.5	−16.5	1.5	−28.5	−13.5	30.5	−20.5	−0.5	18.5	14.5	−28.5	−0.5	−14.5	18.5	−0.5	28.5	13.5
46	26.5	6.5	19.5	15.5	24.5	3.5	17.5	10.5	−31.5	−4.5	−22.5	−13.5	−15.5	18.5	−4.5	30.5	−26.5	−14.5	−3.5	16.5	5.5	−30.5	−11.5	16.5	11.5	−16.5	5.5	−30.5	−10.5
47	16.5	11.5	−30.5	−5.5	−29.5	−1.5	−20.5	−8.5	−8.5	20.5	−1.5	29.5	−24.5	−11.5	−1.5	21.5	31.5	5.5	20.5	9.5	11.5	−16.5	5.5	−30.5	−5.5	30.5	11.5	−16.5	−13.5
48	−22.5	−13.5	31.5	4.5	−12.5	23.5	−2.5	25.5	−2.5	25.5	12.5	−23.5	29.5	9.5	4.5	−23.5	8.5	−21.5	3.5	−25.5	28.5	0.5	18.5	14.5	0.5	−28.5	−14.5	18.5	10.5
49	−2.5	25.5	12.5	−23.5	29.5	1.5	20.5	8.5	8.5	−20.5	1.5	−29.5	−18.5	−7.5	14.5	−27.5	14.5	−27.5	18.5	7.5	3.5	−24.5	−10.5	17.5	−4.5	31.5	13.5	−22.5	27.5
50	4.5	−31.5	−13.5	22.5	12.5	−23.5	2.5	−25.5	2.5	−25.5	−12.5	23.5	23.5	5.5	−11.5	25.5	25.5	11.5	5.5	−23.5	20.5	8.5	−29.5	−1.5	1.5	−29.5	−8.5	20.5	−28.5
51	14.5	−18.5	0.5	−28.5	−9.5	21.5	−7.5	27.5	21.5	9.5	−27.5	−7.5	0.5	−28.5	−14.5	18.5	−28.5	−0.5	−18.5	−14.5	26.5	6.5	19.5	15.5	−15.5	19.5	−6.5	26.5	−27.5
52	−8.5	20.5	−1.5	29.5	−24.5	−3.5	−17.5	−10.5	31.5	4.5	22.5	13.5	−5.5	30.5	11.5	−16.5	−11.5	16.5	−5.5	30.5	13.5	−22.5	4.5	−31.5	10.5	−17.5	3.5	−24.5	28.5
53	18.5	14.5	−28.5	−0.5	27.5	7.5	21.5	9.5	−7.5	27.5	9.5	−21.5	−9.5	28.5	−21.5	−0.5	21.5	0.5	−9.5	28.5	−27.5	−7.5	−21.5	−9.5	7.5	−27.5	−9.5	21.5	6.5
54	−20.5	−8.5	29.5	1.5	10.5	−17.5	3.5	−24.5	−13.5	22.5	−4.5	31.5	12.5	−30.5	16.5	2.5	2.5	−16.5	−30.5	−12.5	−12.5	23.5	−2.5	25.5	−2.5	25.5	12.5	−23.5	−1.5
55	−30.5	−5.5	−16.5	−11.5	−15.5	19.5	−6.5	26.5	−26.5	−6.5	−19.5	−15.5	27.5	7.5	21.5	9.5	−7.5	27.5	9.5	−21.5	−2.5	25.5	12.5	−23.5	12.5	−23.5	2.5	−25.5	−6.5
56	24.5	3.5	17.5	10.5	−30.5	−5.5	−16.5	−11.5	−16.5	−11.5	30.5	5.5	−30.5	−5.5	−16.5	−11.5	−16.5	−11.5	30.5	5.5	−21.5	−9.5	27.5	7.5	−9.5	21.5	−7.5	27.5	1.5
57	−21.5	−9.5	27.5	7.5	18.5	14.5	−28.5	−0.5	−14.5	18.5	−0.5	28.5	−0.5	20.5	30.5	13.5	28.5	1.5	16.5	10.5	−5.5	30.5	11.5	−16.5	−11.5	16.5	−5.5	30.5	21.5
58	19.5	15.5	−26.5	−6.5	3.5	−24.5	−10.5	17.5	−4.5	31.5	13.5	−22.5	5.5	−22.5	−27.5	−15.5	11.5	−17.5	7.5	−26.5	−18.5	−14.5	28.5	0.5	14.5	−18.5	0.5	−28.5	−18.5
59	25.5	2.5	23.5	12.5	−6.5	26.5	15.5	−19.5	−19.5	−15.5	26.5	6.5	18.5	15.5	−30.5	−4.5	−14.5	26.5	−16.5	−3.5	−28.5	−0.5	−18.5	−14.5	−0.5	28.5	14.5	−18.5	−21.5
60	−31.5	−4.5	−22.5	−13.5	−23.5	−12.5	25.5	2.5	−25.5	−2.5	−23.5	−12.5	−23.5	−13.5	27.5	6.5	−25.5	−10.5	−7.5	19.5	−11.5	16.5	−5.5	30.5	5.5	−30.5	−11.5	16.5	18.5
61	5.5	−30.5	−11.5	16.5	20.5	8.5	−29.5	−1.5	1.5	−29.5	−8.5	20.5	−27.5	−15.5	−5.5	22.5	7.5	−26.5	−11.5	17.5	29.5	1.5	20.5	8.5	8.5	−20.5	1.5	−29.5	8.5
62	−3.5	24.5	10.5	−17.5	5.5	−30.5	−11.5	16.5	11.5	−16.5	5.5	−30.5	30.5	13.5	0.5	−20.5	16.5	10.5	−28.5	−1.5	10.5	−17.5	3.5	−24.5	−13.5	22.5	−4.5	31.5	−15.5
63	−9.5	21.5	−7.5	27.5	−0.5	28.5	14.5	−18.5	28.5	0.5	18.5	14.5	9.5	−20.5	5.5	−31.5	−21.5	−1.5	11.5	−24.5	4.5	−31.5	−13.5	22.5	3.5	−24.5	−10.5	17.5	−8.5
64	15.5	−19.5	6.5	−26.5	−17.5	−10.5	24.5	3.5	22.5	13.5	−31.5	−4.5	−12.5	22.5	−0.5	29.5	−2.5	17.5	28.5	8.5	19.5	15.5	−26.5	−6.5	−6.5	26.5	15.5	−19.5	15.5

A.2. Proof of Proposition 1

(i) First we prove $((d+1)^{a_t} D_i \ \vdots \ \alpha)$ is a p^{d+1} full factorial design. Since D_i is a p^d full factorial design, so is $(d+1)^{a_t} D_i$. Suppose $D_i = (\gamma_1, \gamma_2, \ldots, \gamma_d)$, then $(d+1)^{a_t} D_i = ((d+1)^{a_t}\gamma_1, (d+1)^{a_t}\gamma_2, \ldots, (d+1)^{a_t}\gamma_d)$, where $\gamma_1, \gamma_2, \ldots, \gamma_d$ are independent columns. As α is any column of $(d+1)^{a_k} D_i$, we can set $\alpha = (d+1)^{a_k}\gamma_l$, where l is some integer of $1, 2, \ldots, d$. It is obvious that α is independent of $(d+1)^{a_t}\gamma_i$ if $i \neq l$. Meanwhile α is also independent of $(d+1)^{a_t}\gamma_l$ since $a_k \neq a_t$. Therefore, α is independent of $(d+1)^{a_t} D_i$. We finish the proof that $((d+1)^{a_t} D_i \ \vdots \ \alpha)$ is a p^{d+1} full factorial design. Similarly we can prove that $((d+1)^{a_k} D_i \ \vdots \ \beta)$ is a p^{d+1} full factorial design.

(ii) Since D_i satisfies Condition 2, the remaining columns of D_i is a p^d full factorial design. According to Proposition 1(i), α and the remaining columns of $(d+1)^{a_t} D_i$ after deleting one column in each $(d+1)^{a_t} D_i$ form a p^{d+1} full factorial design.

A.3. Proof of Theorem 1

First, we prove Z^* is a LHD. As the grouping method satisfies Condition 2 and Proposition 1(i), the group is a p^{d+v} full factorial design with the $d+v-1$ columns from one part and one column from the other part. From the first 2^{u+v-1} columns of R_{uv}, we can know that the first 2^{u+v-1} columns are signed permutation of $p, p^2, \ldots, p^{2^{u+v-1}}$ and the elements of the second 2^{u+v-1} columns only have one ± 1. It is similar for the second 2^{u+v-1} columns of R_{uv}. According to the algorithm of matrix multiplication, the element $-(p-1)/2, \ldots, (p-1)/2$ can be extended to the $-(p^d-1)/2, -(p^d-3)/2, \ldots, (p^d-1)/2$ after rotating. So the matrix Z^* is a LHD. Next, we prove the orthogonality of Z^*. $(D^{(1)^*}, D^{(2)^*}, \ldots, D^{(s_v)^*})$ is the part of orthogonal array and the R_{uv} is the rotation matrix. Both of them are all column-orthogonal. So for any two columns of Z^*, it is orthogonal (see Tables 4–8).

Table 4
The grouping of $OA(32, 31, 2, 2)$.

Group	columns			
D_1	1	2	3	4
D_2	15	25	35	45
D_3	12	23	34	124
D_4	125	235	345	1245
D_5	13	24	123	234
D_6	135	245	1235	2345

Table 5
The grouping of $OA(27, 13, 3, 2)$.

Group			column	
D_1	1	2	13	23
D_2	13^2	23^2	12	12^2
D_3	123	12^23	123^2	12^23^2

Table 6
The grouping of $OA(81, 40, 3, 2)$.

Group		columns		
D_1	1	2	3	4
D_2	14^2	1^224	12^234^2	$1^223^24^2$
D_3	1^22^23	2^23^24	13^24	124
D_4	1234^2	1^2234^2	$1^22^234^2$	$1^22^23^24^2$
D_5	$1^22^23^2$	$2^23^24^2$	1^23^2	2^24^2
D_6	1^23^24	12^24	123^24^2	1^223
D_7	2^234	13^2	24^2	1^234
D_8	12^2	23^2	34^2	1^24^2
D_9	1^22^24	$12^23^24^2$	1^223^2	2^234^2
D_{10}	$1^23^24^2$	1^22^2	2^23^2	3^24^2

A.4. Proof of Theorem 3

By the rotation matrices R_{uv} and S_{uv}, any two distinct columns $(f^{(1)}(d_1), f^{(1)}(d_2)) = (e_1, e_2)$ form an $OA(p^{d+v}, 2, p, 2)$. So (i) holds. If considering any three distinct columns d_1, d_2, d_3, we have $(f^{(1)}(d_1), f^{(1)}(d_2), f^{(1)}(d_3)) = (e_1, e_2, e_3)$. When the three columns from the same group, the (e_1, e_2, e_3) is the $OA(p^{d+v}, 3, p, 3)$. According to Lemma 4, (ii) holds. Then we consider any two columns d_1 and d_2 which can be expressed as (11). We collapse the p^{d+v} levels of d_1 into the p levels by mapping $f^{(1)}(d_1)$ in (12) and collapse the p^{d+v} levels of d_2 into the p^2 levels by $f^{(2)}(d_2)$, then we have $f^{(2)}(d_2) = pe_2^{(1)} + e_2^{(2)}$. So $(f^{(1)}(d_1), f^{(2)}(d_2) = (e_1^{(1)}, pe_2^{(1)} + e_2^{(2)})$. If d_1 and d_2 are from the same group, $(e_1^{(1)}, pe_2^{(1)} + e_2^{(2)})$ can be a $OA(p^{d+v}, 2, p \times p^2, 2)$ by taking any two columns from $Z_i{}^*$ corresponding to R_{uv} or S_{uv} except column pairs $(2i-1, 2i)$, for $i = 1, 2, \ldots, 2^{u+v-1}$. By calculating, we obtain $\binom{2^{u+v}}{2} - 2^{u+v-1} = 2^{u+v}(2^{u+v-1} - 1)$ column pairs by R_{uv} or S_{uv}, which achieve a stratification on a $p \times p^2$ or $p^2 \times p$ grid. According to the property of R_{uv}, such $2^{u+v}(2^{u+v-1} - 1)$ column pairs by R_{uv} (but for S_{uv} is not true) can also achieve stratification on a $p^2 \times p^2$ grid.

A.5. Proof of Theorem 4

We need to prove $(p^2e_1^{(1)} + pe_1^{(2)} + e_1^{(3)}, p^2e_2^{(1)} + pe_2^{(2)} + e_2^{(3)})$ is an $OA(p^{d+v}, 2, p^3 \times p^3, 2)$. By Lemma 3, if $(e_1^{(1)}, e_1^{(2)}, e_1^{(3)}, e_2^{(1)}, e_2^{(2)}, e_2^{(3)})$ is an $OA(p^{d+v}, 6, p, 6)$, then the proof is accomplished. Since the grouping of the design D_i satisfying Condition 2, it is easy to know that if $e_1^{(1)}, e_1^{(2)}, e_1^{(3)}$ and $e_2^{(1)}, e_2^{(2)}, e_2^{(3)}$ are from different part B_{ij} in the same group D_i, then such six columns form an $OA(p^{d+v}, 6, p, 6)$. In other words, the columns in $Z_i{}^*$ obtained by the column pairs in each $B_{i,j}$ cannot achieve the stratifications on a $p^3 \times p^3$ if the columns in Z^* obtained by the same $B_{i,j}$, because each $B_{i,j}$ has strength 3. There are 4 columns in each $B_{i,j}$ and $j = 1, \ldots, 2^{u+v-2}$. The number of such column pairs is $\binom{4}{2}2^{u+v-2}$. As the total column pairs in D_i is $\binom{2^{u+v}}{2}$, so we have $\binom{2^{u+v}}{2} - \binom{4}{2}2^{u+v-2} = 2^{u+v+1}(2^{u+v-2} - 1)$ column pairs which can achieve the stratifications on $p^3 \times p^3$ grids. The proof for the stratifications on $p^2 \times p^2$ grids is similar, we omit it here.

Table 7
The grouping of $OA(243, 121, 3, 2)$.

Group	Group
$D_{1,1} = (5D_1 \ \vdots \ 5^2 D_1)$	$D_{6,1} = (D_6 \ \vdots \ 5D_6)$
$D_{2,1} = (5D_2 \ \vdots \ 5^2 D_2)$	$D_{7,1} = (D_7 \ \vdots \ 5^2 D_7)$
$D_{3,1} = (D_3 \ \vdots \ 5^2 D_3)$	$D_{8,1} = (D_8 \ \vdots \ 5D_8)$
$D_{4,1} = (D_4 \ \vdots \ 5D_4)$	$D_{9,1} = (5D_9 \ \vdots \ 5^2 D_9)$
$D_{5,1} = (D_5 \ \vdots \ 5D_5)$	$D_{10,1} = (D_{10} \ \vdots \ 5^2 D_{10})$

Note: D_i is as in Table 9, for $i = 1, \ldots, 10$.

Table 8
The grouping of $OA(128, 127, 2, 2)$.

Group	columns															
D_1	1	2	3	123	4	5	6	456	17	27	37	1237	47	57	67	4567
D_2	12	23	34	14	45	56	126	124	127	237	347	147	457	567	1267	1247
D_3	13	24	35	1245	46	125	236	1345	137	247	357	12 457	467	1257	2367	13 457
D_4	1234	2345	3456	1346	12 456	1356	146	1236	12 347	23 457	34 567	13 467	124 567	13 567	1467	12 367
D_5	15	26	235	136	234	345	2356	36	157	267	2357	1367	2347	3457	23 567	367
D_6	1256	346	2456	12 356	25	12 346	145	256	12 567	3467	24 567	123 567	257	123 467	1457	2567
D_7	134	356	1235	2346	1246	246	23 456	123 456	1347	3567	12 357	23 467	12 467	2467	234 567	1234567

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